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# Motion on the $n$ -dimensional ellipsoid under the influence of a harmonic force revisited

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## Abstract

The  $n$  integrals in involution for the motion on the  $n$ -dimensional ellipsoid under the influence of a harmonic force are explicitly found. The classical separation of variables is given by the inverse momentum map. In the quantum case the Schrödinger equation separates into one-dimensional equations that coincide with those obtained from the classical separation of variables. We show that there is a more general orthogonal parametrization *à la* Jacobi that depends on two arbitrary real parameters. Also, if there is a certain relation between the spring constants and the ellipsoid semiaxes, the motion under the influence of such a harmonic potential is equivalent to the free motion on the ellipsoid.

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## 1. Introduction

In this paper we are concerned with the motion on a  $n$ -dimensional ellipsoid under the influence of a harmonic potential. The problem was first posed by Jacobi (1884) in the nineteenth century in the context of explicitly solvable by quadratures differential equations most of them originating in completely integrable Hamiltonian systems. In the last decades the problem was considered mostly by mathematicians obtaining new results including a description of integrals in involution and connection with hyperelliptic curves of genus  $n$  (see Moser (1980a, 1980b), Uhlenbeck (1982) and Mumford (1984)). However, in this approach no explicit separation of variables was found. Quite recently we have obtained a new form for the  $n$  prime integrals in involution for the free motion on the ellipsoid in the Jacobi parametrization (Diță 1999), a form which is very convenient for proving the separation of variables and solving the Hamilton–Jacobi equation.

The purpose of this paper is to extend our previous results in two directions: to find a generalization of the Jacobi orthogonal parametrization and the most general form of the

harmonic potential for which the motion is still completely integrable. As unexpected results we obtained a new orthogonal parametrization that depends on two arbitrary real parameters generalizing the Jacobi one, and if the spring constants  $k_1$  and  $k_2$  along two axes of symmetry satisfy the relation  $a_1 k_1 = a_2 k_2$ , where  $a_1$  and  $a_2$  are the squares of the corresponding semi-axes, then the motion under the influence of such a harmonic potential is the free motion on the ellipsoid. In our approach the separation of variables is realized by the inverse of the momentum map that provides an explicit factorization into Liouville tori. In the standard approach the separation of variables is a rather difficult problem, (see e.g. Kalnis *et al* (1994)) where the separated variables are defined as zeros of the off diagonal elements of the associated Lax matrix.

The paper is organized as follows: in section 2 we define a two-dimensional family of Jacobi elliptic coordinates and in section 3 we prove explicitly their orthogonality. The integrals in involution for the free motion on the ellipsoid are found in section 4 and we show that they are a particular case of a more general form. The most general form of the harmonic potential for which the motion is still integrable is given in section 5 and the classical separation of variables is obtained in section 6. The separation of the Schrödinger equation is shown in section 7 and the paper ends with a conclusion. In the appendix are collected a few known properties of the Vandermonde determinant which are used in the paper.

## 2. Generalized Jacobi's coordinates on the $n$ -dimensional ellipsoid

Usually the  $n$ -dimensional ellipsoid is viewed as a surface immersed in a  $(n + 1)$ -dimensional Euclidean space, defined by the equation

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_{n+1}^2}{a_{n+1}} = 1$$

where  $a_i, i = 1, \dots, n + 1$  are positive numbers,  $a_i \in \mathbf{R}_+$ , and we suppose that they are ordered such that  $a_1 > \dots > a_{n+1} > 0$ . We describe the  $n$ -dimensional ellipsoid by intrinsic coordinates and here we slightly generalize Jacobi's approach by defining the generalized elliptic coordinates as the solutions of the equation

$$\sum_{i=1}^{n+1} \frac{b_i x_i^2}{a_i(b_i - \lambda)} = 1 \quad (1)$$

with  $b_i \in \mathbf{R}^*$ ,  $b_i \neq b_j$ , for  $i \neq j, i, j = 1, \dots, n + 1$ . We rewrite (1) in the form

$$1 + \sum_{i=1}^{n+1} \frac{b_i x_i^2}{a_i(\lambda - b_i)} = \frac{\lambda Q(\lambda)}{P(\lambda)} \quad (2)$$

where  $P(\lambda)$  and  $Q(\lambda)$  are the monic polynomials

$$Q(\lambda) = \prod_{i=1}^n (\lambda - u_i) \quad P(\lambda) = \prod_{i=1}^{n+1} (\lambda - b_i).$$

Here  $u_i, i = 1, \dots, n$  are the Jacobi coordinates and the factor  $\lambda$  comes from the fact that  $\lambda = 0$  is a solution of (1) when the coordinates  $x_i$  lie on the ellipsoid.

Calculating the residues on both left- and right-hand sides of equation (2) one gets

$$x_i^2 = a_i \frac{Q(b_i)}{P'(b_i)} \quad i = 1, \dots, n + 1. \quad (3)$$

If  $b_i = a_i, i = 1, \dots, n + 1$ , equation (3) gives the usual Jacobi parametrization of the  $n$ -dimensional ellipsoid. For  $a_i = r^2, i = 1, \dots, n + 1$ , the relation (3) gives the orthogonal parametrization of the  $n$ -dimensional sphere of radius  $r$  and in this case (1) has the form

$$\sum_{i=1}^{n+1} \frac{x_i^2}{\lambda - b_i} = 0.$$

Our approach is sufficiently general and gives the parametrization of any quadric defined by  $a_i \in \mathbf{R}^*$ .

Parametrization (3) depends on  $2n + 2$  parameters, half of them  $b_i \in \mathbf{R}^*, i = 1, \dots, n + 1$ , being arbitrary real numbers and for this reason the corresponding Jacobi coordinates  $u_i$  could not be orthogonal. However, there does exist an orthogonal parametrization, different from the usual one, which depends on  $n + 3$  parameters. In other words there is more than one orthogonal system of coordinates of Jacobi type and this situation could be interpreted as a hidden symmetry of the problem. We found that this symmetry is two-dimensional and the independent parameters in equation (3) leading to orthogonal coordinates may be chosen as  $a_i, i = 1, \dots, n + 1, b_1$  and  $b_2$ , the last two being arbitrary non-zero real numbers with  $b_1 \neq b_2$ .

To find the new parametrization we deduce from (3)

$$\frac{\dot{x}_i}{x_i} = \frac{1}{2} \sum_{l=1}^n \frac{\dot{u}_l}{u_l - b_i}$$

where a dot means the derivative with respect to time, and

$$\sum_{i=1}^{n+1} \dot{x}_i^2 = \frac{1}{4} \sum_{l,m=1}^n \dot{u}_l \dot{u}_m \sum_{i=1}^{n+1} \frac{x_i^2}{(u_l - b_i)(u_m - b_i)}.$$

Taking into account relation (3), a careful inspection shows that the function

$$c_{lm} = \sum_{i=1}^{n+1} \frac{x_i^2}{(u_l - b_i)(u_m - b_i)} \quad l \neq m$$

is a symmetric polynomial with respect to  $u_i$  in its  $n - 2$  variables, having degree equal to  $n - 2$  and  $n - 1$  independent coefficients. An important property is that these coefficients do not depend on  $l$  and  $m$ , i.e. for all  $l \neq m, c_{lm}$  defines a single function; instead of  $n(n - 1)/2$  different polynomials we have only one. Imposing now the condition  $c_{lm} \equiv 0$  for  $l \neq m$  we obtain  $n - 1$  equations which can be solved with respect to  $b_i$ . So the number of parameters of the new parametrization is  $2n + 2 - (n - 1) = n + 3$ .

To see what happens let us consider the case  $n = 2$ ; in this case the condition  $c_{12} \equiv 0$  is equivalent to the relation

$$b_3 = \frac{a_1 b_2 - a_2 b_1 + a_3(b_1 - b_2)}{a_1 - a_2}.$$

By iteration we obtain for  $n = 4$  the additional constraint

$$b_4 = \frac{a_1 b_2 - a_2 b_1 + a_4(b_1 - b_2)}{a_1 - a_2}.$$

Thus we suppose that the general case is given by

$$b_n = \frac{a_1 b_2 - a_2 b_1 + a_n(b_1 - b_2)}{a_1 - a_2} \quad n = 3, \dots, n + 1.$$

We introduce now a uniform notation by defining two new parameters

$$\alpha = \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} \quad \text{and} \quad \beta = \frac{b_1 - b_2}{a_1 - a_2}.$$

Solving with respect to  $b_1$  and  $b_2$  we find

$$b_i = \alpha + \beta a_i \quad i = 1, \dots, n+1. \quad (4)$$

In the following section we prove that the two-dimensional family of Jacobi coordinates whose parametrization is given by relation (3) with  $b_i$  as in (4) is orthogonal for all real values of  $\alpha \in \mathbf{R}$  and  $\beta \in \mathbf{R}^*$ .

### 3. Orthogonality property

For proving the orthogonality of the new coordinates we write equation (2) in another form

$$\sum_{i=1}^{n+1} \frac{x_i^2}{a_i(\lambda - b_i)} = \frac{Q(\lambda)}{P(\lambda)}.$$

In the last relation we substitute  $b_i = \alpha + \beta a_i$ ,  $i = 1, \dots, n+1$ , and after minor transformation we get

$$\sum_{i=1}^{n+1} \frac{x_i^2}{\alpha + \beta a_i - \lambda} = \frac{1}{\beta} \left( 1 - \frac{(\lambda - \alpha)Q(\lambda)}{P(\lambda)} \right). \quad (5)$$

For  $l \neq m$  we have

$$c_{lm} \sum_{i=1}^{n+1} \frac{x_i^2}{(u_l - b_i)(u_m - b_i)} = \frac{1}{u_l - u_m} \left( \sum_{i=1}^{n+1} \frac{x_i^2}{u_m - b_i} - \sum_{i=1}^{n+1} \frac{x_i^2}{u_l - b_i} \right).$$

Substituting  $b_i = a + \beta a_i$  in the last relation and using (5) and the property  $Q(u_i) = 0$ ,  $i = 1, \dots, n$ , we find that  $c_{lm} = 0$  for  $l \neq m$ , i.e. the new coordinates are orthogonal, and our assumption expressed by the relation (4) is true.

For  $l = m$  we have

$$c_{ll} = \sum_{i=1}^{n+1} \frac{x_i^2}{(u_l - b_i)^2} = - \frac{d}{dz} \sum_{i=1}^{n+1} \frac{x_i^2}{(z - b_i)} \Big|_{z=u_l} \quad (6)$$

and we need a calculation of the last sum. It can be obtained by differentiating (5) with respect to  $\lambda$ . We use this result to write the Lagrangian in the form

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n+1} \dot{x}_i^2 = \sum_{j=1}^n g_{jj} \dot{u}_j^2 \quad (7)$$

where  $g_{jj} = -\frac{1}{4\beta} \frac{(u_j - \alpha)Q'(u_j)}{P(u_j)}$  are the components of the diagonal metric tensor and  $Q'(u_j) = dQ(x)/dx|_{x=u_j}$ . For  $\alpha = 0$  and  $\beta = 1$  one recovers the usual result (Moser 1980b).

Following the standard procedure we find the Hamiltonian of the free motion on the  $n$ -dimensional ellipsoid

$$\mathcal{H} = \sum_{j=1}^n p_j u_j - \mathcal{L} = -2\beta \sum_{j=1}^n g^{jj} p_j^2 \quad (8)$$

where  $g^{ii} = P(u_i)/(u_i - \alpha)Q'(u_i)$  and  $u_j, p_j$  are canonical coordinates.

Unlike the classical result we have obtained that the Hamiltonian of the geodesic motion is not uniquely defined, it depends continuously and non-trivially on two arbitrary real parameters  $\alpha$  and  $\beta$ . By changing these parameters one changes the classical state of the system if the

latter one is defined as a point in the phase space but the form of the energy does not change. This property could be interpreted as a gauge symmetry of the classical Jacobi problem. Under the transformations

$$x_i^2 \rightarrow a_i \frac{Q(b_i)}{P'(b_i)} \quad b_i \rightarrow \alpha + \beta a_i \quad \alpha \in \mathbf{R} \quad \beta \in \mathbf{R}^*$$

the Lagrangian  $\mathcal{L}$  and the Hamiltonian  $\mathcal{H}$  remain invariant, and *ipso facto* the equations of motion. Since with given initial conditions the physical motion is unique the only freedom we have in proving the uniqueness is the reparametrization of time. Generally speaking this fact gives rise to a new problem, namely that of finding all the metrics which lead to the same physical motion; in other words how large the hidden symmetry is, or how many geometries describe the same physical process. A solution to this problem could be of interest in the study of more complicated models arising in classical and quantized field theories.

#### 4. Integrals in involution

We introduce now  $n$  integrals in involution that are linearly independent. With this aim we define the symmetric functions of the polynomials  $Q'(u_j) \equiv Q_j(u_j)$

$$Q_j(u_j) = \sum_{k=0}^{n-1} u_j^k S_{n-k-1}^{(j)} \quad (9)$$

where  $S_k^{(j)} = (-1)^k \sigma_k(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n)$  and  $\sigma_k$  is the symmetric polynomial of degree  $k$ . By construction the coordinate  $u_j$  does not enter the symmetric sum  $S_k^{(j)}$ ,  $k = 0, 1, \dots, n-1$ . The following functions:

$$H_k = \sum_{l=1}^n S_{k-1}^{(l)} g^{ll} p_l^2 \quad k = 1, \dots, n \quad (10)$$

with  $H_1 = -\mathcal{H}$  up to a numerical factor are  $n$  integrals in involution for the geodesic motion on the ellipsoid.

The integrals of motion  $H_k$ ,  $k = 1, \dots, n$  in the above form were found by us in Diță (1999); for another approach see e.g. Moser (1980a).

Taking into account the explicit form of the inverse of the metrical tensor  $g^{ii}$  an inspection of (10) shows that for each degree of freedom the contribution to the Hamiltonian  $H_k$  is given by a product of two factors. The first one, the ‘kinematical’ factor, depends on a special structure, in our case the Vandermonde structure defined by the ratio  $f_1(u_1, \dots, u_n) = S_{k-1}^{(i)} / Q'(u_i)$ , and the second one, the kinetic energy,  $f_2(u_i, p_i) = p_i^2 P(u_i) / (u_i - \alpha)$  depends on the ‘physics’, in our case the geometry of the body. The important issue is the factorization  $f_1(u_1, \dots, u_n) \cdot f_2(u_i, p_i)$  where  $f_1$  has no momentum dependence and  $f_2$  depends only on a pair of canonical variables and nothing else.

Let  $g(p, u) = \mathcal{H}(p, u)$  be an arbitrary function depending on the canonical variables  $p$  and  $u$  which is invertible with respect to the momentum  $p$ . As we will see later the invertibility condition is necessary for the separation of variables in the Hamilton–Jacobi equation. In particular we may suppose that  $\mathcal{H}(p, u)$  is a one-dimensional Hamiltonian. For each  $n \in \mathbf{N}$  we define an  $n$ -dimensional integrable model by giving its  $n$  integrals in involution

$$\mathbf{H}_k(p, u) = \sum_{i=1}^n \frac{S_{k-1}^{(i)}}{Q'(u_i)} g(p_i, u_i) \quad k = 1, \dots, n. \quad (11)$$

We will prove the involution property in the more general case (11). We have

$$\begin{aligned}
\{\mathbf{H}_k, \mathbf{H}_l\} &= \sum_{j=1}^n \left( \frac{\partial \mathbf{H}_k}{\partial u_j} \frac{\partial \mathbf{H}_l}{\partial p_j} - \frac{\partial \mathbf{H}_k}{\partial p_j} \frac{\partial \mathbf{H}_l}{\partial u_j} \right) \\
&= \sum_{j=1}^n \frac{\partial g(p_j, u_j)}{\partial p_j} \left( \frac{S_{l-1}^{(j)}}{Q'(u_j)} \frac{\partial \mathbf{H}_k}{\partial u_j} - \frac{S_{k-1}^{(j)}}{Q'(u_j)} \frac{\partial \mathbf{H}_l}{\partial u_j} \right) \\
&= \sum_{j=1}^n \sum_{i=1}^n \frac{1}{Q'(u_j)} \frac{\partial g(p_j, u_j)}{\partial p_j} \left( S_{l-1}^{(j)} \frac{\partial}{\partial u_j} \left( g(p_i, u_i) \frac{S_{k-1}^{(i)}}{Q'(u_i)} \right) \right. \\
&\quad \left. - S_{k-1}^{(j)} \frac{\partial}{\partial u_j} \left( g(p_i, u_i) \frac{S_{l-1}^{(i)}}{Q'(u_i)} \right) \right) \\
&= \sum_{j=1}^n \sum_{i=1}^n \frac{1}{Q'(u_j)} \frac{\partial g(p_j, u_j)}{\partial p_j} \frac{\partial}{\partial u_j} \left( \frac{g(p_i, u_i) (S_{k-1}^{(i)} S_{l-1}^{(j)} - S_{k-1}^{(j)} S_{l-1}^{(i)})}{Q'(u_i)} \right).
\end{aligned}$$

The last step was possible because the symmetric functions  $S_k^{(j)}$  and  $S_l^{(j)}$  do not depend on  $u_j$ . Looking at the last expression it is easily seen that the partial derivative with respect to  $u_j$  vanishes for  $i = j$ . For  $i \neq j$  we have to show that

$$\frac{\partial}{\partial u_j} \frac{S_{k-1}^{(i)} S_{l-1}^{(j)} - S_{k-1}^{(j)} S_{l-1}^{(i)}}{u_i - u_j} = 0$$

but this is a consequence of the following identities:

$$\frac{\partial}{\partial u_j} S_{k-1}^{(i)} = -S_{k-2}^{(i,j)} \quad \text{and} \quad S_{k-1}^{(i)} - S_{k-1}^{(j)} = (u_i - u_j) S_{k-2}^{(i,j)}$$

where the upper index  $(i, j)$  means that the corresponding expression does not depend on both  $u_i$  and  $u_j$ . In this way we have shown that  $\{\mathbf{H}_k, \mathbf{H}_l\} = 0$ .

## 5. Harmonic potential

In the following we want to find the most general form of the harmonic potential for which the motion on the ellipsoid under the influence of this potential is still integrable. Moser (1980a) says that ‘the motion on an ellipsoid under the influence of a potential  $|x|^2$  is also integrable [and] this was shown already by Jacobi’. On the other hand Arnold *et al* (1993) make a stronger statement: ‘Jacobi showed that the problem of free motion on an ellipsoid remains integrable if the point is subjected to the action of an elastic force whose direction passes through the centre of the ellipsoid’, which might be understood as suggesting that the spring constants on different axes are different. We start with the most general form for the harmonic potential

$$\mathcal{U} = \frac{1}{2} \sum_{i=1}^{n+1} k_i x_i^2$$

with  $k_i \neq k_j$  for  $i \neq j$  and we look for conditions on  $k_i$  for which the motion is integrable. We show that the most general form for  $\mathcal{U}$  depends on the semiaxes of the ellipsoid and two arbitrary parameters which can be taken as  $k_1$  and  $k_2$ . By substitution of the relation (3) in the above formula we get

$$\mathcal{U} = \frac{1}{2} \sum_{i=1}^{n+1} k_i x_i^2 = \sum_{k=0}^n \left( \sum_{i=1}^{n+1} \frac{k_i a_i}{P'(b_i)} b_i^k \right) S_{n-k}$$

where  $S_k$  are up to a ‘ $\pm$ ’ sign the symmetric functions of  $u_1, \dots, u_n$ . This means that  $\mathcal{U}$  is a polynomial of degree  $n$  in the elliptic coordinates defined by equations (3), (4). The motion described by the Hamiltonian  $H = \mathcal{H} + \mathcal{U}$  is completely integrable iff the coefficients of all the products  $u_i, \dots, u_l$  vanish such that  $\mathcal{U}$  in the new variables should have the form  $\mathcal{U} = A + B(u_1 + \dots + u_n)$ . The vanishing of these coefficients leads to the relations

$$k_i = \frac{a_1 a_2 (k_1 - k_2) - a_i (a_1 k_1 - a_2 k_2)}{a_i (a_2 - a_1)} \quad i = 3, 4, \dots, n + 1$$

which shows that the spring constants do not depend on the previous introduced parameters  $\alpha$  and  $\beta$  as should be the case. As in the preceding case we define two new parameters

$$\gamma = \frac{a_1 a_2 (k_1 - k_2)}{a_2 - a_1} \quad \text{and} \quad \delta = \frac{a_2 k_2 - a_1 k_1}{a_2 - a_1} \quad \gamma, \delta \in \mathbf{R}$$

such that  $k_i$  has the dependence

$$k_i(a_i) = \frac{\gamma + \delta a_i}{a_i} \quad i = 1, \dots, n + 1$$

that can be viewed as the action of a special element of  $GL_2^+(\mathbf{R})$  which transforms the right half-space into itself. For  $\gamma = 0$  one recovers the classical result. Thus we obtained that there is a two-dimensional family of coefficients  $k_i$ , which depend also on the semiaxes of the ellipsoid, for which the motion is integrable. Making all the calculation one finds

$$\mathcal{U} = \frac{1}{2} \left( n \frac{\alpha \delta}{\beta} + \gamma + \delta (a_1 + a_2) + \sum_{k=3}^{n+1} a_k - \frac{\delta}{\beta} \sum_{i=1}^n u_i \right)$$

which is the most general form of the harmonic potential for which the motion is completely integrable. If  $\delta = 0$  or  $a_1 k_1 = a_2 k_2$  the harmonic potential in elliptic coordinates reduces to a constant, i.e. the motion is the free geodesic motion, a fact which is noted for the first time.

### 6. Separation of variables

The Hamiltonian of the problem is  $H = \mathcal{H} + \mathcal{U}$  and we do not yet know how the other prime integrals look. We shall neglect the constant terms appearing in  $\mathcal{U}$  such that

$$H = -2\beta \sum_{i=1}^n \frac{P(u_i)}{(u_i - \alpha) Q'(u_i)} p_i^2 - \frac{\delta}{2\beta} \sum_{i=1}^n u_i. \tag{12}$$

The Hamiltonians  $H_k$  (see relations (10)) depend on the symmetric functions  $S_k^{(j)}$ . A similar sum also appears in the potential, namely  $S_1 = -\sum_{i=1}^n u_i$ , where  $S_k$  are similarly defined as in equation (9) by the relation

$$Q(x) = \prod_{i=1}^n (x - u_i) = \sum_{k=0}^n x^k S_{n-k}. \tag{13}$$

It is easily seen that  $dS_k/du_j = -S_{k-1}^{(j)}$ . For the other prime integrals we define the potentials  $\mathcal{U}_k = \frac{\delta}{2\beta} S_k$  for  $k = 2, \dots, n$  such that the integrals in involution are given by

$$\mathcal{H}_k = H_k + \mathcal{U}_k = -2\beta \sum_{i=1}^n \frac{S_{k-1}^{(i)} P(u_i)}{(u_i - \alpha) Q'(u_i)} p_i^2 + \frac{\delta}{2\beta} S_k \quad k = 1, \dots, n. \tag{14}$$

The involution property follows straightforwardly

$$\begin{aligned} \{\mathcal{H}_i, \mathcal{H}_j\} &= \{H_i, H_j\} + \{H_i, \mathcal{U}_j\} + \{\mathcal{U}_i, H_j\} + \{\mathcal{U}_i, \mathcal{U}_j\} \\ &= \sum_{i=1}^n \left( \frac{\partial \mathcal{U}_i}{\partial u_l} \frac{\partial \mathcal{H}_j}{\partial p_l} - \frac{\partial \mathcal{H}_i}{\partial p_l} \frac{\partial \mathcal{U}_j}{\partial u_l} \right) \\ &= -2\delta \sum_{l=1}^n \frac{P(u_l) p_l}{(u_l - \alpha) Q'(u_l)} (S_{i-1}^{(l)} S_{j-1}^{(l)} - S_{j-1}^{(l)} S_{i-1}^{(l)}) = 0. \end{aligned}$$

The next important point is the separation of variables.

Let  $M^{2n} \simeq T^*(\mathbf{R}^n)$  be the canonically symplectic phase space of the dynamical system defined by the Hamilton functions (14). We define the momentum map by

$$\varepsilon : M^{2n} \rightarrow \mathbf{R}^n : M_{\mathbf{h}} = \{(u_i, p_i) : \mathcal{H}_i = -h_i, i = 1, \dots, n\} \quad h_i \in \mathbf{R}. \tag{15}$$

This application is such that  $\varepsilon^{-1}(M_{\mathbf{h}})$  realizes the separation of variables giving an explicit factorization of Liouville’s tori into one-dimensional ovals. Our goal is to construct explicitly the application  $\varepsilon^{-1}(M_{\mathbf{h}})$  and to do that we write system (14) in matrix form. With the notation  $f(u, p) = P(u)p^2/(u - \alpha)$  (14) is written as

$$\begin{aligned} 2\beta \begin{pmatrix} S_0^{(1)}/Q'(u_1) & S_0^{(2)}/Q'(u_2) & \dots & S_0^{(n)}/Q'(u_n) \\ S_1^{(1)}/Q'(u_1) & S_1^{(2)}/Q'(u_2) & \dots & S_1^{(n)}/Q'(u_n) \\ \dots & \dots & \dots & \dots \\ S_{n-1}^{(1)}/Q'(u_1) & S_{n-1}^{(2)}/Q'(u_2) & \dots & S_{n-1}^{(n)}/Q'(u_n) \end{pmatrix} \begin{pmatrix} f(u_1, p_1) \\ f(u_2, p_2) \\ \dots \\ f(u_n, p_n) \end{pmatrix} - \frac{\delta}{2\beta} \begin{pmatrix} S_1 \\ S_2 \\ \dots \\ h_n \end{pmatrix} \\ = \begin{pmatrix} h_1 \\ h_2 \\ \dots \\ h_n \end{pmatrix}. \end{aligned}$$

Multiplying to the left by the matrix

$$V = \begin{pmatrix} u_1^{n-1} & u_1^{n-2} & \dots & 1 \\ u_2^{n-1} & u_2^{n-2} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ u_n^{n-1} & u_n^{n-2} & \dots & 1 \end{pmatrix} \tag{16}$$

we get a diagonal matrix with its non-zero elements equal to unity that multiplies the vector column  $(f(u_1, p_1), f(u_2, p_2), \dots, f(u_n, p_n))^t$ , where t means transpose. For a proof, see relation (A.2) given in the appendix. The solution is

$$f(u_i, p_i) = -\frac{\delta}{4\beta^2} u_i^n + \frac{1}{2\beta} \sum_{k=1}^n h_k u_i^{n-k} \quad i = 1, \dots, n \tag{17}$$

where the first term on the right-hand side is a result of the property  $Q(u_i) = 0$  (see equation (13)) written in the form  $u_i^n = -\sum_{k=0}^{n-1} u_i^k S_{n-k}$ . The above relations have the classical form (Sklyanin 1995)

$$\varphi(x_i, p_i, h_1, \dots, h_n) = 0 \quad i = 1, \dots, n$$

which for  $h_i = c_i, i = 1, \dots, n$  give an explicit parametrization of Liouville’s tori.

Taking into account the form of  $f(u_i, p_i)$  and defining  $R(u_i) = (u_i - \alpha)(-\frac{\delta}{4\beta^2} u_i^n + \frac{1}{2\beta} \sum_{k=1}^n h_k u_i^{n-k})$  we get

$$p_i = \epsilon_i \sqrt{\frac{R(u_i)}{P(u_i)}} \quad i = 1, \dots, n$$

where  $\epsilon_i = \pm 1$ . By the substitution  $p_i \rightarrow \partial S/\partial t_i$  the Hamilton–Jacobi equation separates and the solution has the form

$$S(\mathbf{h}, u_1, \dots, u_n) = \sum_{i=1}^n \epsilon_i \int_{u_i^0}^{u_i} \sqrt{\frac{R(w)}{P(w)}} dw.$$

On the last expression one can see that all the subtleties of the problem are encoded by the hyperelliptic curve  $y^2 = P(x)R(x)$  whose genus equals  $n$ .

The motion described by the prime integrals (11) is also separable and

$$g(u_i, p_i) = \sum_{k=0}^{n-1} h_{n-k} u_i^k \quad i = 1, \dots, n.$$

To obtain the geodesic equations  $g(u, p)$  has to be invertible with respect to  $p$ . With the notation  $\mathcal{R}(u) = \sum_{k=0}^{n-1} h_{n-k} u^k$  the momentum is given by

$$p_i = g^{-1}(\mathcal{R}(u_i))$$

where  $g^{-1}$  denotes the inverse of  $g$  with respect to  $p$  and the solution of the Hamilton–Jacobi equation now has the form

$$S(\mathbf{h}, u_1, \dots, u_n) = \sum_{i=1}^n \epsilon_i \int_{u_i^0}^{u_i} g^{-1}(\mathcal{R}(w)) dw.$$

The above formulae allow us to choose new canonical variables (Arnold 1978), and in the last case we may take  $\mathcal{Q}_k = \mathcal{H}_k, k = 1, \dots, n$  and the canonically conjugated variables  $\mathcal{P}_i, i = 1, \dots, n$ . The Hamilton equations are

$$\dot{\mathcal{Q}}_i = 0 \quad i = 1, \dots, n \quad \dot{\mathcal{P}}_1 = -1 \quad \dot{\mathcal{P}}_i = 0 \quad i = 2, \dots, n$$

and therefore  $\mathcal{Q}_i = h_i, i = 1, \dots, n$  and  $\mathcal{P}_1 = -t + g_1, \mathcal{P}_k = g_k, k = 2, \dots, n$  with  $g_i, h_i \in \mathbf{R}, i = 1, \dots, n$ . Because

$$\mathcal{P}_i = -\frac{\partial S}{\partial \mathcal{Q}_i} = -\frac{\partial S}{\partial h_i} = -\sum_{i=1}^n \int_{u_i^0}^{u_i} (g^{-1})'(\mathcal{R}(w)) w^{n-i} dw$$

where  $(g^{-1})'(z) = dg^{-1}(z)/dz$  one obtains the system of equations

$$-t\delta_{1,j} + b_j = \sum_{i=1}^n \int_{u_i^0}^{u_i} (g^{-1})'(\mathcal{R}(t)) t^{n-j} dt \quad j = 1, \dots, n$$

which represents the implicit form of the geodesics, and shows that the canonical equations are integrable by quadratures. In the above formulae we singled out the first prime integral  $\mathcal{H}_1$ ; if we start with  $\mathcal{H}_k$  as the Hamiltonian then the change in the above formulae is  $\delta_{1j} \rightarrow \delta_{kj}$ .

### 7. Quantization

To begin, we consider the Hamiltonian  $\mathcal{H}_1$  given by equation (14). It is well known that because of the ambiguities concerning the ordering of  $u$  and  $p$  we must use the Laplace–Beltrami operator (Podolsky 1928). Its general form is  $\Delta = \frac{1}{\sqrt{g}}(\sqrt{g}g^{ij}p_j), i, j = 1, \dots, n$ , where  $g = \det(g_{ij})$  and  $g_{ij}$  is the metric tensor. In our case  $g_{ij} = -\frac{1}{4\beta} \frac{(u_i - \alpha)Q'(u_i)}{P(u_i)} \delta_{ij}$  and  $g^{ii} = P(u_i)/(u_i - \alpha)Q'(u_i)$ . Let  $V_n$  denote the determinant of matrix  $V^t$ , where  $V^t$  means the transpose of  $V$  (see equation (16)) and  $V_{n-1}^{(j)}$  the determinant of the matrix obtained by

removing from  $V^t$  the last row and the  $j$ th column. Using relations (A.3) and (A.4) from the appendix we find that up to an inessential numerical factor

$$g = V_n^2 \prod_{i=1}^n \frac{u_i - \alpha}{P(u_i)}$$

and using that, the Schrödinger equation generated by  $\mathcal{H}_1$  is written, after some simplification, in the form

$$2\beta \sum_{i=1}^n \frac{1}{V_n} \sqrt{\frac{P(u_i)}{u_i - \alpha}} \frac{\partial}{\partial u_i} \left( (-1)^{n-i} V_{n-1}^{(i)} \sqrt{\frac{P(u_i)}{u_i - \alpha}} \frac{\partial \Psi}{\partial u_i} \right) - \left( \frac{\delta}{2\beta} \sum_{i=1}^n u_i \right) \Psi = E_1 \Psi.$$

Since the factor  $V_{n-1}^{(i)}$  does not depend on  $u_i$  it can be pulled out of the bracket and the preceding equation takes the form

$$\sum_{i=1}^n (-1)^{n-i} V_{n-1}^{(i)} \sqrt{\frac{P(u_i)}{u_i - \alpha}} \frac{\partial}{\partial u_i} \left( \sqrt{\frac{P(u_i)}{u_i - \alpha}} \frac{\partial \Psi}{\partial u_i} \right) - \left( \frac{\delta}{4\beta^2} V_n \sum_{i=1}^n u_i \right) \Psi = \frac{E_1}{2\beta} V_n \Psi.$$

Now we use the Jacobi identities (A.5) and find

$$\sum_{i=1}^n (-1)^{n-i} V_{n-1}^{(i)} \left[ \sqrt{\frac{P(u_i)}{u_i - \alpha}} \frac{\partial}{\partial u_i} \left( \sqrt{\frac{P(u_i)}{u_i - \alpha}} \frac{\partial \Psi}{\partial u_i} \right) + \left( -\frac{\delta}{4\beta^2} u_i^n + \sum_{k=0}^{n-1} c_{n-k} u_i^k \right) \Psi \right] = 0$$

which is equivalent to  $n$  equations of the form

$$\sqrt{\frac{P(u_i)}{u_i - \alpha}} \frac{\partial}{\partial u_i} \left( \sqrt{\frac{P(u_i)}{u_i - \alpha}} \frac{\partial \Psi_i}{\partial u_i} \right) + \left( -\frac{\delta}{4\beta^2} u_i^n + \sum_{k=0}^{n-1} c_{n-k} u_i^k \right) \Psi_i = 0 \quad i = 1, \dots, n. \quad (18)$$

Here  $c_1 = -E_1/2\beta$  and the other  $c_k$  are arbitrary.

The direct approach, starting with equation (17), is simpler, the problem being one-dimensional, and one gets the same equation (18). It has the advantage that the arbitrary coefficients  $c_k$  are identified with  $c_k = -h_k/2\beta$ , i.e.  $c_k$  are the eigenvalues of the Hamiltonians  $\mathcal{H}_k$ .

In this way the solving of the Schrödinger equation was reduced to the solving of a Sturm–Liouville equation whose general form is

$$-\frac{d}{dx} \left( p(x) \frac{df(x)}{dx} \right) + v(x)f(x) = \lambda r(x)f(x)$$

and the above equation has to be resolved on an interval  $[a, b]$ . It is well known that its eigenfunctions will live in a Hilbert space iff  $p(x)r(x) > 0$  on  $[a, b]$ . If  $p(x)$  has a continuous first derivative and  $p(x)r(x)$  a continuous second derivative then by making the following coordinate and function transforms:

$$\varphi = \int^u \left( \frac{r(x)}{p(x)} \right)^{1/2} dx \quad \Phi = (r(u)p(u))^{1/4} f(u) \quad (19)$$

one brings the preceding equation to the standard form

$$-\frac{d^2 \Phi}{d\varphi^2} + q(\varphi)\Phi = \lambda \Phi$$

where

$$q(\varphi) = \frac{\mu''(\varphi)}{\mu(\varphi)} - \frac{v(u)}{r(u)} \quad \mu(\varphi) = (p(u)r(u))^{1/4}$$

and  $u = u(\varphi)$  is the solution of the inverse Abel problem (19).

In our case, equation (18), the transformation is

$$\varphi = \int_{u_0}^u \left( \frac{R(u)}{P(u)} \right)^{1/2} du$$

and the Schrödinger equation gets

$$-\frac{d^2\Phi}{d\varphi^2} + \frac{\mu''(\varphi)}{\mu(\varphi)}\Phi = h_1\Phi \quad (20)$$

where  $\mu(\varphi) = (R(u(\varphi)))^{1/4}$  and in  $R(u)$  we made the rescaling  $h_k \rightarrow h_k/h_1$ ,  $k = 1, \dots, n$ , i.e. the solving of (18) is equivalent to solving equation (20) which represents the motion of a one-dimensional particle in the potential generated by  $R(u(\varphi))$ .

For  $n = 1$  and  $\delta = 0$  relation (20) is nothing else than the equation for the one-dimensional rotator

$$\frac{d^2\Psi}{d\varphi^2} + l^2\Psi = 0$$

with the solution  $\Psi(\varphi) = \frac{1}{\sqrt{2\pi}} e^{il\varphi}$ ,  $l \in \mathbb{Z}$ , etc. In all the other cases we have to make use of the theory of hyperelliptic curves and/or  $\theta$ -functions in order to obtain explicit solutions. This problem will be treated elsewhere.

## 8. Conclusion

In this paper we have obtained the most general form of the harmonic potential for which the motion of a point on the  $n$ -dimensional ellipsoid is completely integrable and we found another form of the integrals in involution. The advantage of our approach is that separation of variables is very easy being an immediate consequence of the Stäckel structure appearing in equations of motion. We have shown that the Vandermonde structure is powerful enough to allow construction of new  $n$ -dimensional completely integrable models. Two such models could be given by the one-dimensional Hamiltonians,  $g(u, p) = (\sin u/u)p^2$  and  $g(u, p) = tgu e^{\alpha p}$  (see (11)). These models are interesting since in the first example  $g(u, p)$  is a function, which has a denumerable number of zeros and the second one has a denumerable number of zeros and poles, in both cases the hyperelliptic curve being of infinite genus. Thus these models show that the dimension  $n$  of the system has no direct connection with the number of zeros and/or poles of the function  $g(u, p)$ .

Other examples of  $n$ -dimensional Hamiltonians are obtained for example from the many-body elliptic Calogero–Moser (CM) model (Braden *et al* 1999) or the elliptic Ruijsenaars model (Ruijsenaars 1987), starting with the one-dimensional Hamiltonians  $H_{\text{CM}}(u, p) = p^2/2 + v^2\mathcal{P}_\tau(u)$  and  $H_R(u, p) = \cosh(\alpha p)\sqrt{1 - 2(\alpha v)^2\mathcal{P}_\tau(u)}$ , respectively, where  $\mathcal{P}_\tau(u)$  is the Weierstrass function, and using the above machinery.

In conclusion we discovered a new method for obtaining  $n$ -dimensional completely integrable systems starting with one-dimensional Hamiltonians. Also interesting is the existence of a class of orthogonal metrics *à la* Jacobi for which the Lagrangian is gauge invariant. This result gives rise to the problem of description of all the metrics that lead to the same physics. A step in this direction could be the revivification of the techniques discovered by Stäckel, Levi-Civita, Painlevé and many others which fell into undeserved oblivion.

## Appendix

In the following we collect a few known properties of the Vandermonde determinant, the novelty being their presentation in the context of algebraic duality. Let  $a_i, i = 1, \dots, n$  be  $n$  real (complex) numbers. We define the polynomial

$$P(x) = \prod_{i=1}^n (x - a_i) = \sum_{k=0}^n S_k x^{n-k} \quad (\text{A.1})$$

where  $S_k = (-1)^k \sigma_k(a_1, \dots, a_n)$  and  $\sigma_k$  denote the elementary symmetric polynomials of degree  $k$ . From (A.1) we have

$$P(a_i) = 0 \quad i = 1, \dots, n$$

i.e. the vectors  $X_1 = (S_0, S_1, \dots, S_n)$  and  $X_2^i = (a^n, \dots, a, 1), i = 1, \dots, n$  are orthogonal  $(X_1, X_2^i) = 0$  under the usual Euclidean scalar product. In order to see a few interesting duality relations we define the polynomials

$$P_j(x) = \frac{P(x)}{x - a_j} = \sum_{k=0}^{n-1} S_k^{(j)} x^{n-k-1} \quad j = 1, \dots, n.$$

From the property  $P_j(a_i) = P'(a_i) \delta_{ij}$  we deduce that the vectors  $U_j = (S_0^{(j)}, \dots, S_{n-1}^{(j)})$  and  $V_i = (a_i^{n-1}, \dots, 1)$  are bi-orthogonal, i.e.

$$(V_j, U_i) = P'(a_i) \delta_{ij} \quad i, j = 1, \dots, n \quad (\text{A.2})$$

showing that  $V_j$  and  $U_i$  are dual to each other. The above relation tells us that the product of the matrices  $U = (U_i)_1^n$  and  $V = (V_j^t)_1^n$  where  $t$  means transpose is a diagonal matrix.

It is easily seen that the Vandermonde determinant has two dual equivalent definitions

$$V_n(a_1, \dots, a_n) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ S_1^{(1)} & S_1^{(2)} & \dots & S_1^{(n)} \\ \dots & \dots & \dots & \dots \\ S_{n-1}^{(1)} & S_{n-1}^{(2)} & \dots & S_{n-1}^{(n)} \end{vmatrix} \quad (\text{A.3})$$

where  $|\cdot|$  means the determinant of the array. Let  $V_{n-1}^{(j)}$  be the Vandermonde determinant obtained by removing the  $j$ th column and the last row of the first array entering (A.3). Then the following relations hold:

$$V_{n-1}^{(j)} = \prod_{\substack{1 \leq k < l \leq n \\ k \neq j, l \neq j}} (a_l - a_k) \quad (\text{A.4})$$

$$\prod_{j=1}^n V_{n-1}^{(j)} = (V_n)^{n-2}$$

$$\frac{V_n}{V_{n-1}^{(j)}} = (-1)^{n-j} P'(a_j) \quad j = 1, \dots, n. \quad (\text{A.5})$$

We give now the most general form of direct and dual Jacobi identities. By replacing the last row of the first form of  $V_n$  by the row  $(a_1^k, \dots, a_n^k)$  and expanding over this row we find the identities (Babelon and Talon 1992)

$$\sum_{i=1}^n (-1)^{n-1} a_i^k V_{n-1}^{(i)} = \begin{cases} 0 & k = 0, 1, \dots, n-2 \\ V_n & k = n-1 \\ V_n \sum_{i=1}^n a_i & k = n. \end{cases} \quad (\text{A.6})$$

By replacing the last row of the dual form of  $V_n$  by  $S_k^{(1)}, \dots, S_k^{(n)}$  we find the dual Jacobi identity

$$\sum_{i=1}^n (-1)^{n-1} S_k^{(i)} V_{n-1}^{(i)} = \begin{cases} 0 & k = 0, \dots, n-2 \\ V_n & k = n-1. \end{cases} \tag{A.7}$$

The above formulae are a consequence of a more general result. Let  $A_{ij}$  be the minor of the  $(i, j)$  element of the determinant  $V_n$ , i.e.

$$A_{ij} = \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ a_1 & \dots & a_{j-1} & a_{j+1} & \dots & a_n \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ a_1^{i-1} & \dots & a_{j-1}^{i-1} & a_{j+1}^{i-1} & \dots & a_n^{i-1} \\ a_1^{i+1} & \dots & a_{j-1}^{i+1} & a_{j+1}^{i+1} & \dots & a_n^{i+1} \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ a_1^{n-1} & \dots & a_{j-1}^{n-1} & a_{j+1}^{n-1} & \dots & a_n^{n-1} \end{vmatrix}.$$

From relation (A.2) we get

$$a_i^{n-1} = - \sum_{k=1}^{n-1} S_k^{(j)} a_i^{n-k-1} \quad i = 1, \dots, n$$

and substitute it in the last row of  $A_{ij}$ . Afterwards we multiply the first row by  $S_{n-1}^{(j)}$ , the second by  $S_{n-2}^{(j)}$ , the  $i$ th by  $S_{n-i}^{(j)}$ , etc, and add them to the last row and we get

$$A_{ij} = \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ a_1 & \dots & a_{j-1} & a_{j+1} & \dots & a_n \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ a_1^{i-1} & \dots & a_{j-1}^{i-1} & a_{j+1}^{i-1} & \dots & a_n^{i-1} \\ a_1^{i+1} & \dots & a_{j-1}^{i+1} & a_{j+1}^{i+1} & \dots & a_n^{i+1} \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ -S_{n-i-1}^{(j)} a_1^i & \dots & -S_{n-i-1}^{(j)} a_{j-1}^i & -S_{n-i-1}^{(j)} a_{j+1}^i & \dots & -S_{n-i-1}^{(j)} a_n^i \end{vmatrix} \\ = (-1)^{n-i-1} S_{n-i-1}^{(j)} V_{n-1}^{(j)}(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n).$$

To obtain a dual result we denote by  $B_{ij}$  the corresponding minor obtained from the dual form of  $V_n$ . As in the preceding case we use the relation

$$S_{n-1}^{(l)} = - \sum_{k=0}^{n-2} S_k^{(l)} a_j^{n-k-1} \quad j = 1, \dots, n$$

and substitute it in the last row of  $B_{ij}$ . Multiplying the first row by  $a_j^{n-1}$ , the second by  $a_j^{n-2}$ , etc, and adding to the last row we find

$$\begin{aligned}
 B_{ij} = & \left( \begin{array}{cccccc}
 1 & \cdots & 1 & & 1 & \cdots & 1 \\
 S_1^{(1)} & \cdots & S_1^{(j-1)} & & S_1^{(j+1)} & \cdots & S_1^{(n)} \\
 \cdot & \cdots & \cdot & & \cdot & \cdots & \cdot \\
 S_{i-1}^{(1)} & \cdots & S_{i-1}^{(j-1)} & & S_{i-1}^{(j+1)} & \cdots & S_{i-1}^{(n)} \\
 S_{i+1}^{(1)} & \cdots & S_{i+1}^{(j-1)} & & S_{i+1}^{(j+1)} & \cdots & S_{i+1}^{(n)} \\
 \cdot & \cdots & \cdot & & \cdot & \cdots & \cdot \\
 -a_j^{n-i-1} S_i^{(1)} & \cdots & -a_j^{n-i-1} S_i^{(j-1)} & & -a_j^{n-i-1} S_i^{(j+1)} & \cdots & -a_j^{n-i-1} S_i^{(n)}
 \end{array} \right) \\
 & = (-a_j)^{n-i-1} V_{n-1}^{(j)}(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n).
 \end{aligned}$$

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